XII-MATHEMATICS: APPLICATIONS OF MATRICES AND DETERMINANTS

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PRE-REQUISITES

- Determinants
- Algebra of matrices
- Non-singular matrices
- Cofactors
- Properties of cofactors
- Elimination method of solving system of equations

Contents of Chapter 1

Outcomes

- Upon completion of this chapter, students will be able to
  - Demonstrate a few fundamental tools for solving systems of linear equations:
    - Adjoint of a square matrix
    - Inverse of a non-singular matrix
    - Elementary row and column operations
    - Row-echelon form
    - Rank of a matrix
  - Use row operations to find the inverse of a non-singular matrix
  - Illustrate the following techniques in solving system of linear equations by
    - Matrix inverse method
    - Cramer's rule
    - Gaussian elimination method
  - Test the consistency of system of non-homogeneous linear equations
  - Test for nontrivial solution of system of homogeneous linear equations

Adjoint of a square matrix $A = [a_{ij}]_{(n\times n)}$

- Minor M_{ij}
- Cofactor $A_{ij} = (-1)^{i+j} M_{ij}$
- An important property connecting the elements of a square matrix and their cofactors is that
  - the sum of the products of the entries (elements) of a row and the corresponding cofactors of the elements of the same row is equal to the determinant of the matrix;
  - and the sum of the products of the entries (elements) of a row and the corresponding cofactors of the elements of any other row is equal to 0.
- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
- $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = |A|$
- $a_{11}A_{21} + a_{12}A_{21} + a_{13}A_{23} = 0$
- $a_{11}A_{31} + a_{12}A_{31} + a_{13}A_{33} = 0$

Adjoint of a square matrix $A = [a_{ij}]_{(n \times n)}$

- The matrix of cofactors of A is defined as the matrix obtained by replacing each element a_{ij} of A with the corresponding cofactor A_{ij}. The adjoint matrix of A is defined as the transpose of the matrix of cofactors of A. It is denoted by $adj(A)$.
- $adj(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T$
- If A is a square matrix of order n, then $A[adj(A)] = [adj(A)]A = |A|I_n$

Inverse of a square matrix

- Let A be a square matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I_n$, then the matrix B is called an inverse matrix of A.
- Inverse is unique.
  - Let there be two inverses B and C of A.
  - Then, by definition, we have $AB = BA = I$ and $AC = CA = I$.
  - Using these equations, we get
    - $B = BI = B(AC) = (BA)C = IC = C$.

  This proves the uniqueness.
- If A is non-singular matrix of order n, then $A\left[\frac{1}{|A|}adj(A)\right] = \left[\frac{1}{|A|}adj(A)\right]A = I_n$
- $\frac{1}{|A|}adj(A)$ is the inverse of A. It is unique and is denoted by A^{-1}.
- $A^{-1} = \frac{1}{|A|}adj(A)$.
- A^{-1}exists if and only if A is non-singular.

Properties of inverses of matrices

- $|A^{-1}| = \frac{1}{|A|}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$, where $\lambda \neq 0$.
- Left Cancellation Law $AB = AC \implies B = C$.
- Right Cancellation Law $BA = CA \implies B = C$.
- Reversal Law for Inverses $(AB)^{-1} = B^{-1}A^{-1}$
- Law of Double Inverse $(A^{-1})^{-1} = A$
- $(adjA)^{-1} = adj(A^{-1}) = \frac{1}{|A|}A$
- $|adj(A)| = |A|^{n-1}$
- $adj(adj(A)) = |A|^{n-2}A$
- $adj(\lambda A) = \lambda^{n-1} adj(A)$, $\lambda \neq 0$
- $|adj(adj(A))| = |A|^{(n-1)^2}$
- $adj(A^T) = (adj(A))^T$

Illustrations for adjoint

- If A is symmetric, prove that $adj(A)$ is also symmetric.
- If A and B are any two non-singular square matrices of order, then
  - $adj(AB) = adj(B)adj(A)$
- If $adj(A) = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}.
  - $A^{-1} = \pm\frac{1}{\sqrt{|adj(A)|}} adj(A) = \pm\frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$
- Find A if $adj(A) = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$
  - $A = \pm\frac{1}{\sqrt{|adj(A)|}} adj(adj(A)) = \pm \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

Application of matrices to Cryptography

- Cryptography is an art of communication between two people by keeping the information not known to others.
- It is based upon two factors, namely encryption and decryption.
  - Encryption means the process of transformation of an information (plain form)into an unreadable form (coded form).
  - Decryption means the transformation of the coded message back into original form.
  - Encryption and decryption require a secret technique which is known only to the sender and the receiver.
  - This secret is called a key.
  - One way of generating a key is by using a non-singular matrix to encrypt a message by the sender.
  - The receiver decodes (decrypts) the message to retrieve the original message by using the inverse of the matrix.
  - The matrix used for encryption is called encryption matrix(encoding matrix).
  - The inverse of the encryption matrix is called decryption matrix (decoding matrix).

Application of matrices to Cryptography

- For simplicity,
  - Suppose that the sender and receiver consider messages in alphabets A-Z only.
  - Both assign the numbers 1-26 to the letters A-Z respectively, and the number 0 to a blank space.
  - The sender employs a key as post-multiplication by a non-singular matrix of order 3 of his own choice.
  - The receiver uses post-multiplication by the inverse of the matrix which has been chosen by the sender.
- Example: "WELCOME"
  - $\begin{bmatrix} 23 & 5 & 12 \end{bmatrix} \begin{bmatrix} 3 & 15 & 13 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}$
  - Encryption matrix $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
  - Encoded message $\begin{bmatrix} 45 & -28 & 23 \end{bmatrix}\begin{bmatrix} 46 & -18 & 3 \end{bmatrix}\begin{bmatrix} 5 & -5 & 5 \end{bmatrix}$
  - Decryption matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

Elementary Transformations of a Matrix

- Elementary Row and Column Operations
  - (i) The interchanging of any two rows (columns) of the matrix.
  - (ii) Replacing a row (column) of the matrix by a scalar multiple of the row (column) by a non-zero scalar.
  - (iii) Replacing a row (column) of the matrix by a sum of the row (column) with a scalar multiple of another row(column) of the matrix.
- We use the following notations for elementary row transformations:
  - (i) Interchanging of $i-$th and $j-$th rows is denoted by $R_i \longleftrightarrow R_j$.
  - (ii) The multiplication of each element of $i-$th row by a nonzero constant λ is denoted by $R_i \longrightarrow R_j$.
  - (iii) Addition to $i-$th row, a nonzero constant λ multiple of $j-$th throw is denoted by $R_i \longrightarrow R_i + \lambda R_j$.
- Two matrices A and B of same order are said to be equivalent to one another if one can be obtained from the other by the applications of elementary transformations. Symbolically, we write $A \sim B$ to mean that the matrix A is equivalent to the matrix B.

Row-Echelon form

- A nonzero matrix E is said to be in a row-echelon form if:
  - All zero rows of E occur below every nonzero row of E.
  - If the first nonzero element in any $i-$th row of E occurs in the $j-$th column of E, then all other entries in the $j-$th column of E below the first nonzero element of $i-$th row are zeros.
  - The first nonzero entry in the $i-$th row of E lies to the left of the first nonzero entry in the $(i+1)-$th row of E.
- A nonzero matrix is in a row-echelon form if all zero rows occur as bottom rows of the matrix, and if the first nonzero element in any lower row occurs to the right of the first nonzero entry in the higher row.
- A row-echelon form of a matrix is not necessarily unique. But, if we make the first non-zero entry in each row of the row-echelon form as 1, then the row-echelon form is unique.

Reduce the matrix $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$ to row-echelon form.

- $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix} \underbrace{R_1 \longleftrightarrow R_2} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 4 & 2 & 0 & 0 \end{bmatrix}$
- $\underbrace{R_3 \longrightarrow R_3 + 4 \times R_1} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 2 & 8 & 20 \end{bmatrix}$
- $\underbrace{R_3 \longrightarrow R_3 - \frac{2}{3} \times R_2} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & \frac{22}{3} & 16 \end{bmatrix}$
- $\underbrace{R_3 \longrightarrow \frac{3}{2} \times R_3} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 11 & 24 \end{bmatrix}$

Carl Friedrich Gauss (1777-1855) was a German mathematician and physicist

Rank of a Matrix

- Let A be a given matrix, not necessarily a square matrix. $+$
- The rank of a matrix A is defined as the order of a highest order non-vanishing minor of the matrix A. It is denoted by the symbol $\rho(A)$. The rank of a zero matrix is defined to be 0.
- If a matrix A contains at-least one non-zero element, then $\rho(A) \geq 1$.
- The rank of the identity matrix I_n is n.
- If the rank of a matrix A is r, then there exists at-least one minor of A of order r which does not vanish and every minor of A of order $r+1$ and higher order (if any) vanishes.
- If A is an $m \times n$ matrix, then $\rho(A) \leq \min\{m, n\}$
- A square matrix A of order n has inverse if and only if $\rho(A) = n$.
- The rank of a matrix in row echelon form is the number of nonzero rows in it.
- The rank of a nonzero matrix is equal to the number of non-zero rows in a row-echelon form of the matrix.

Find the rank of $A = \begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$.

- $A = \begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix} \underbrace{\begin{matrix} R_2 \longrightarrow 4 \times R_2, \\ R_3 \longrightarrow 4 \times R_3 \end{matrix}} \begin{bmatrix} 4 & 3 & 1 & -2 \\ -12 & -4 & -8 & 16 \\ 24 & 28 & -4 & 8 \end{bmatrix}$
- $\underbrace{\begin{matrix} R_2 \longrightarrow R_2 + 3 \times R_1, \\ R_3 \longrightarrow R_3 - 6 \times R_1 \end{matrix}} \begin{bmatrix} 4 & 3 & 1 & -2 \\ 0 & 5 & -5 & 10 \\ 0 & 10 & -10 & 20 \end{bmatrix}$
- $\underbrace{R_3 \longrightarrow R_3 - 2 \times R_2} \begin{bmatrix} 4 & 3 & 1 & -2 \\ 0 & 5 & -5 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $\rho(A) = 2$

Elementary matrices and their uses

- An elementary matrix is defined as a matrix which is obtained from an identity matrix by applying only one elementary transformation.
- If we are dealing with matrices with three rows, then all elementary matrices are square matrices of order 3 which are obtained by carrying out only one elementary row operations on the unit matrix I_3. Every elementary row operation that is carried out on a given matrix A can be obtained by pre-multiplying A with elementary matrix.
- For instance, consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
- Suppose that we do the elementary transformation $R_2 \longrightarrow R_2 + \lambda R_3$ on A where $\lambda \neq 0$ is a constant.

Elementary matrices and their uses

- Then, we get

$$A \underbrace{R_2 \longrightarrow R_2 + \lambda R_3} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + \lambda a_{31} & a_{22} + \lambda a_{32} & a_{23} + \lambda a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{R_2 \longrightarrow R_2 + \lambda R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix}$
- It is an elementary matrix (say E).
- Pre-multiplying A by E, we get

$$EA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + \lambda a_{31} & a_{22} + \lambda a_{32} & a_{23} + \lambda a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- So, the elementary transformation $R_2 \longrightarrow R_2 + \lambda R_3$ on A is the same as that of pre-multiplying the matrix A by the elementary matrix E.

Elementary matrices and their uses

- Similarly, we can show that
  - the elementary transformation $R_1 \longleftrightarrow R_3$ on A is the same as that of pre-multiplying the matrix A by the elementary matrix
    - $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.
  - the elementary transformation $R_3 \longleftrightarrow \lambda \times R_3$ on A is the same as that of pre-multiplying the matrix A by the elementary matrix
    - $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$.
- Theorem 1.11 Every non-singular matrix can be transformed to an identity matrix, by a sequence of elementary row operations.

- Example 1.19 Using Gauss-Jordan method, find the inverse of $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.
- Step 1: Augment I_3 on the right-side of A to get $[A|I_3]$:
  - $[A|I_3] = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$.
- Step 2: Apply elementary transformatons on $[A|I_3]$ to get $[I_3|A^{-1}]$:
  - $[I_3|A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$.
- $\therefore A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

Applications of Matrices: Solving System of Linear Equations

- Matrix form $AX = B$
- A system of linear equations having at least one solution is said to be consistent.
- A system of linear equations having no solution is said to be inconsistent.
- If the number of the equations of a system of linear equations is equal to the number of unknowns of the system, then the coefficient matrix of the system is a square matrix. Further, if A is a non- singular square matrix, then the system of equations can be solved by any one of the following methods
  - Matrix inversion method
  - Cramer's rule
  - Gaussian elimination method

Solving System of Linear Equations by Matrix inversion method

- Matrix inversion method
  - $X = A^{-1}B$
- Solve the following system:
  - $2x + 3y - z = 9, x + y + z = 9, 3x - y - z + 1 = 0$
- $AX = B$ where
  - $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$
- $A^{-1} = \frac{1}{16} \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$
- $X = A^{-1}B = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

- Cramer's rule
  - $x_1 = \frac{\Delta_1}{\Delta}, x_2 = \frac{\Delta_2}{\Delta}, x_3 = \frac{\Delta_3}{\Delta}$
- A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-3, 30), (-1, 2)$ and $(4, 2)$. He wants to meet his friend at $(5, 6)$. Will he meet his friend? (Use Cramer's rule to solve the problem).
  - $(-3, 30) \implies 9a - 3b + c = 30$
  - $(-1, 2) \implies a - b + c = 2$
  - $(4, 2) \implies 16a + 4b + c = 2$
- $\Delta = -70, \Delta_1 = -140, \Delta_2 = 420, \Delta_3 = 420.$
- $a = 2, b = -6, c = -6$
- $y = 2x^2 - 6x - 6$
- $(5, 6)$ satisfies the equation.

Solving System of Linear Equations by Gaussian elimination method

- Gaussian elimination method
  - By elementary transformations, bring $[A|B]$ to the form $[E|C]$, where E is in row-echelon form.
  - Apply back-substitution to get the solution.
- This method can be applied even if the coefficient matrix is not a square matrix.

Example for Gaussian elimination method

- Example: The upward velocity $v(t)$ of a rocket over the time interval $2 \leq t \leq 12$ is approximated by a quadratic expression $v(t) = at^2 + bt + c$, where a, b, and c are constants. It has been found that the velocity at times $t = 3, t = 6$, and $t = 9$ seconds are respectively, 64, 133 and 208 miles per second respectively. Find the velocity at time $t = 15$ seconds. (Use Gaussian elimination method to solve the problem.)
  - $v(3) = 64 \implies 9a + 3b + c = 64$
  - $v(6) = 133 \implies 36a + 6b + c = 133$
  - $v(9) = 208 \implies 81a + 9b + c = 208$
- $[A|B] = \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 36 & 6 & 1 & 133 \\ 81 & 9 & 1 & 208 \end{array}\right] \implies \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 0 & 1 & 1 \end{array}\right]$
- $\implies c = 1, b = 20, a = \frac{1}{3}$
- $\therefore v(t) = \frac{t^2}{3} + 20t + 1$
- $\therefore v(15) = 376.$

Consistency of system of linear equations by rank method

- We have two classes of systems of linear equations
  - Non-homogeneous system of linear equations
  - Homogeneous system of linear equations
- Theorem 1.12 (Rouche'-Capelli Theorem)
  - A system of linear equations, written in the matrix form as $AX = B$ is consistent if and only if $\rho(A) = \rho([A|B])$.
- Importance of Rouche'-Capelli theorem
  - If there are n unknowns in the system of equations and $\rho(A) = \rho([A|B]) = n$, then the system $AX = B$ is consistent and has a unique solution.
  - If there are n unknowns in the system and $\rho(A) = \rho([A|B]) = n - k$, $k \neq 0$, then the system is consistent and has infinitely many solutions and these solutions form a $k-$ parameter family. In particular, if there are 3 unknowns in a system of equations and $\rho(A) = \rho([A|B]) = 2$, then the system has infinitely many solutions and these solutions form a one parameter family. In the same manner, if there are 3 unknowns in a system of equations and $\rho(A) = \rho([A|B]) = 1$, then the system has infinitely many solutions and these solutions form a two parameter family.
  - If $\rho(A) \neq \rho([A|B])$, then the system is inconsistent and has no solution.

Example 1.29: Test the consistency, and if possible solve the following system of linear equations: $x+2y-z=3$, $3x-y+2z=1$, $x-2y+3z=3$, $x-y+z+1=0$

- $[A|B] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} \underbrace{\begin{matrix} R_2 \longrightarrow R_2 - 3 \times R_1, \\ R_3 \longrightarrow R_3 - R_1, \\ R_4 \longrightarrow R_4 - R_1 \end{matrix}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -4 & 4 & 0 \\ 0 & -3 & 2 & -4 \end{bmatrix}$
- $\underbrace{R_2 \longrightarrow (1/(-7)) \times R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & (-5/7) & (8/7) \\ 0 & 4 & -4 & 0 \\ 0 & 3 & -2 & 4 \end{bmatrix} \underbrace{\begin{matrix} R_3 \longrightarrow 7 \times R_3, \\ R_4 \longrightarrow 7 \times R_4 \end{matrix}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & (-5/7) & (8/7) \\ 0 & 28 & -28 & 0 \\ 0 & 21 & -14 & 28 \end{bmatrix}$
- $\underbrace{\begin{matrix} R_3 \longrightarrow R_3 - 28 \times R_2, \\ R_4 \longrightarrow R_4 - 21 \times R_2 \end{matrix}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & (-5/7) & (8/7) \\ 0 & 0 & -8 & -32 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

  $\underbrace{R_3 \longrightarrow (1/(-8)) \times R_3} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & (-5/7) & (8/7) \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$
- $\underbrace{R_4 \longrightarrow R_4 - R_3} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & (-5/7) & (8/7) \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longleftrightarrow \begin{matrix} x+2y-z=3, \\ y+(-5/7)z=(8/7) \\ z=4, \\ 0=0 \end{matrix} \Longleftrightarrow \begin{matrix} x=-1, \\ y=4 \\ z=4, \\ 0=0 \end{matrix}$
- $\rho(A) = \rho([A|B]) = 3$
- The system is consistent and the solution is unique.

Example 1.30: Test the consistency, and if possible solve the following system of linear equations: $2x - y + 3z = 4$, $x + y - 3z = -1$, $x - 2y + 6z = 5$, $5x - y + 3z = 7$

- $[A|B] = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 1 & 1 & -3 & -1 \\ 1 & -2 & 6 & 5 \\ 5 & -1 & 3 & 7 \end{bmatrix} \underbrace{R_1 \longrightarrow (1/2) \times R_1} \begin{bmatrix} 1 & -(1/2) & (3/2) & 2 \\ 1 & 1 & -3 & -1 \\ 1 & -2 & 6 & 5 \\ 5 & -1 & 3 & 7 \end{bmatrix}$
- $\underbrace{\begin{array}{c} R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - R_1 \\ R_4 \longrightarrow R_4 - 5 \times R_1 \end{array}} \begin{bmatrix} 1 & -(1/2) & (3/2) & 2 \\ 0 & (3/2) & -(9/2) & -3 \\ 0 & -(3/2) & (9/2) & 3 \\ 0 & (3/2) & -(9/2) & -3 \end{bmatrix}$
- $\underbrace{R_2 \longrightarrow (2/3) \times R_2} \begin{bmatrix} 1 & -(1/2) & (3/2) & 2 \\ 0 & 1 & -3 & -2 \\ 0 & -(3/2) & (9/2) & 3 \\ 0 & (3/2) & -(9/2) & -3 \end{bmatrix}$
- $\underbrace{\begin{array}{c} R_3 \longrightarrow R_3 + (3/2) \times R_2, \\ R_4 \longrightarrow R_4 - (3/2) \times R_2 \end{array}} \begin{bmatrix} 1 & -(1/2) & (3/2) & 2 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Example continued

- $\rho(A) = \rho([A|B]) = 2$
- The system is consistent.
- But, the number of unknowns $= 3$ and so, the solution is not unique.
- Writing down the equations with the echelon form, we get
  - $\begin{cases} x - (1/2)y + (3/2)z = 2, \\ y - 3z = -2, \\ 0 = 0, \\ 0 = 0 \end{cases}$
  - $\begin{cases} z = t \in \mathbb{R} \\ \implies y = -2 + 3t \\ \implies x = -2 + (1/2)(2 + 3t) - (3/2)t = -1 \end{cases}$
- The solution set is a one-parameter family of solutions.
- Here, the given system is consistent and has infinitely many solutions which form a one parameter family of solutions.
- In the above example, the coefficient matrix A is singular.
- So, neither matrix inversion method nor Cramer's rule can be applied.
- However, Gaussian elimination method is applicable and we are able to decide whether the system is consistent or not.
- This example glorifies the supremacy of Gaussian elimination method over other methods.

Example 1.32: Test the consistency, and if possible solve the following system of linear equations: $x - y + z = -9$, $2x - y + z = 4$, $3x - y + z = 6$, $4x - y + 2z = 7$

- $[A|B] = \begin{bmatrix} 1 & -1 & 1 & -9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{bmatrix}$
- $\underbrace{\begin{matrix} R_2 \longrightarrow R_2 - 2 \times R_1 \\ R_3 \longrightarrow R_3 - 3 \times R_1 \\ R_4 \longrightarrow R_4 - 4 \times R_1 \end{matrix}} \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \end{bmatrix}$
- $\underbrace{\begin{matrix} R_3 \longrightarrow R_3 - 2 \times R_2, \\ R_4 \longrightarrow R_4 - 3 \times R_2 \end{matrix}} \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 1 & -23 \end{bmatrix}$
- $\underbrace{R_3 \longleftrightarrow R_4} \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -23 \\ 0 & 0 & 0 & -11 \end{bmatrix}$
- $\rho(A) = 3$ and $\rho([A|B]) = 4 \implies \rho(A) \neq \rho([A|B])$
- $\therefore$ the given system is consistent and has no solution.
- You can also see this by writing down the equations with the echelon form,
  - $\begin{cases} x - y + z = -9, \\ y - z = 22, \\ z = -23, \\ 0 = -11 \end{cases}$
  - $0 = -11$ is a contradiction.

Example 1.33: Find the condition on a, b and c so that the following system of linear equations has one parameter family of solutions: $x+y+z=a$, $x+2y+3z=b$, $3x+5y+7z=c$

- $[A|B] = \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 3 & 5 & 7 & c \end{bmatrix}$
- $\underbrace{\begin{matrix} R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - 3 \times R_1 \end{matrix}} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b-a \\ 0 & 2 & 4 & c-3a \end{bmatrix}$
- $\underbrace{R_3 \longrightarrow R_3 - 2 \times R_2} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b-a \\ 0 & 0 & 0 & c-2b-a \end{bmatrix}$
- There are 3 unknowns.
- For one-parameter family, the condition is
  - $\rho(A) = \rho([A|B]) = 2$
- $\therefore c - 2b - a = 0 \Longrightarrow c = a + 2b$

Homogeneous system of linear equations

- $AX = O$ where O is the zero column matrix.
- It always has the trivial solution.
- So, it is always consistent.
- Since the last column of $[A|B]$ is the zero column matrix, we always have
  - $\rho(A) = \rho([A|B])$
- Suppose there are n unknowns.
- Suppose there are more number of equations than the number of unknowns. Reducing the system by elementary transformations.
- If $\rho(A) = \rho([A|B]) = n$, then $\det(A) \neq 0$ and the system has trivial solution only.
- If $\rho(A) = \rho([A|B]) = n - k, k \neq 0$ then the system has $k-$parameter family of non-trivial solutions.
- If $\rho(A) < n$, then $\det(A) = 0$.
- A homogeneous system $AX = O$ has a non-trivial solution if and only if $\det(A) = 0$.

Example 1.37: Solve the following system of linear equations: $x + y - 2z = 0$, $2x - 3y + z = 0$, $3x - 7y + 10z = 0$, $6x - 9y + 10z = 0$

- $[A|B] = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 2 & -3 & 1 & 0 \\ 3 & -7 & 10 & 0 \\ 6 & -9 & 10 & 0 \end{bmatrix}$
- $\underbrace{\begin{array}{l} R_2 \longrightarrow R_2 - 2 \times R_1 \\ R_3 \longrightarrow R_3 - 3 \times R_1 \\ R_4 \longrightarrow R_4 - 6 \times R_1 \end{array}} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -10 & 16 & 0 \\ 0 & -15 & 22 & 0 \end{bmatrix}$
- $\underbrace{\begin{array}{l} R_2 \longrightarrow (-1/5) \times R_2 \\ R_3 \longrightarrow (-1/2) \times R_3 \end{array}} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 5 & -8 & 0 \\ 0 & -15 & 22 & 0 \end{bmatrix} \underbrace{\begin{array}{l} R_3 \longrightarrow R_3 - 5 \times R_2 \\ R_4 \longrightarrow R_4 + 15 \times R_2 \end{array}} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix}$
- $\underbrace{R_3 \longrightarrow (-1/3) \times R_3} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} \underbrace{R_4 \longrightarrow R_4 - 7 \times R_3} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- There are 3 unknowns and we get
  - $\rho(A) = \rho([A|B]) = 3$
- $\therefore$ the system has trivial solution only.

Exercise 1.7 (3): By using Gaussian elimination method, balance the chemical reaction equation: $C_2H_6 + O_2 \longrightarrow H_2O + CO_2$

- $x_1 \times C_2H_6 + x_2 \times O_2 = x_3 \times H_2O + x_4 \times CO_2$
  - $C \implies 2x_1 - x_4 = 0$
  - $H \implies 6x_1 - 2x_3 = 0$
  - $O \implies 2x_2 - x_3 - 2x_4 = 0$
- $[A|B] = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 6 & 0 & -2 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{bmatrix}$
- $\underbrace{R_1 \longrightarrow (1/2) \times R_1} \begin{bmatrix} 1 & 0 & 0 & -(1/2) & 0 \\ 6 & 0 & -2 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{bmatrix}$
- $\underbrace{R_2 \longrightarrow R_2 - 6 \times R_1} \begin{bmatrix} 1 & 0 & 0 & -(1/2) & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{bmatrix}$
- $\underbrace{R_2 \longleftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & -(1/2) & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 0 & -2 & 3 & 0 \end{bmatrix}$
- $\underbrace{\begin{matrix} R_2 \longrightarrow (1/2) \times R_2 \\ R_3 \longrightarrow -(1/2) \times R_2 \end{matrix}} \begin{bmatrix} 1 & 0 & 0 & -(1/2) & 0 \\ 0 & 1 & -(1/2) & -1 & 0 \\ 0 & 0 & 1 & -(3/2) & 0 \end{bmatrix}$
- $\begin{matrix} x_1 - (1/2)x_4 = 0, \\ x_2 - (1/2)x_3 - x_4 = 0, \\ x_3 - (3/2)x_4 = 0 \end{matrix}$
- $x_4 = t \implies x_3 = (3/2)t, x_2 = x_2 = (7/4)t, x_1 = (1/2)t$
- $t = 4 \implies x_1 = 2, x_2 = 7, x_3 = 6, x_4 = 4. \implies 2C_2H_6 + 7O_2 \longrightarrow 6H_2O + 4CO_2$